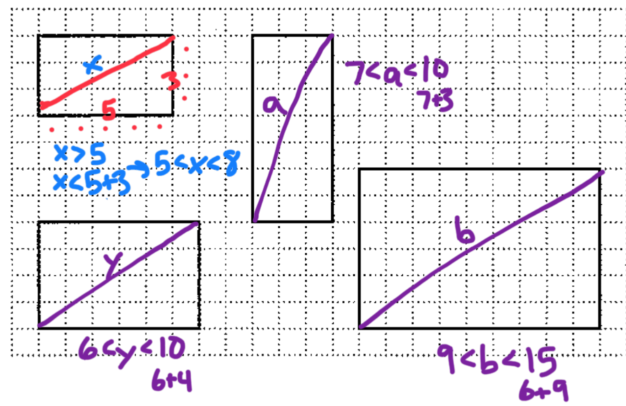
2.15 Three of a Kind: Pythagorean Theorem

Explore

1. For each rectangle shown here, draw a diagonal and estimate its length as accurately as you can. Assume each segment of the grid is one unit. Describe any relationship you see between the lengths of the sides of a rectangle and the length of its diagonal.



Discover

In the *Explore*, you could only *estimate* the lengths of the diagonals of the squares, since the diagonals were not horizontal or vertical segments whose lengths could be easily determined using the grid. There are situations in which we need to be able to find a diagonal length more exactly. This section will introduce you to a formula to do this.

As you work through this section, think about these questions:

1. When can you use the Pythagorean theorem?
2. Is it more difficult to solve the Pythagorean theorem for the hypotenuse or for one of the legs of a right triangle?

When a diagonal is drawn in a square, two right triangles are formed. The length of the diagonal can be determined by using a well-known relationship between the lengths of the sides of a right triangle.

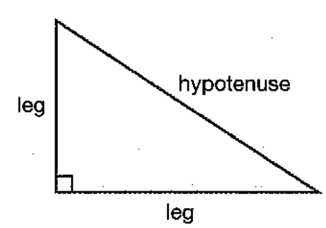
LOOK IT UP

Pythagorean Theorem

A triangle that has one 90° angle is known as a right triangle. In a right triangle, the sides that form the right angle are known as the **legs**. The side across from the right angle is called the **hypotenuse** and is the longest side in a right triangle.

Right triangles have a special relationship between the lengths of their sides that is expressed in the **Pythagorean theorem**, which says

$$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$$



LOOK IT UP

Pythagorean Theorem *(continued)*

The Pythagorean theorem is an example of a **quadratic equation**, since it involves polynomials of the second degree, in which the variable is squared. The Pythagorean theorem is also commonly written as

$$a^2 + b^2 = c^2$$

where a and b represent the lengths of the legs and c represents the length of the hypotenuse.

FOR EXAMPLE, in a right triangle whose legs have lengths of 3 cm and 4 cm, the hypotenuse is 5 cm since

$$\begin{aligned} 3^2 + 4^2 &= 9 + 16 \\ &= 25 \\ &= 5^2 \end{aligned}$$

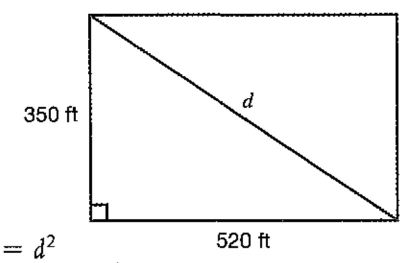
Let's see how the Pythagorean theorem can be used to find a distance.

EXAMPLE 1

At John's school, there is a rectangular patch of grass called the *quad* that measures 350 feet by 520 feet. John needs to walk from one corner to the corner diagonally opposite him. He could walk along the sidewalks that line the perimeter of the quad, or he can cut across the quad in a straight line. He knows that the shortest distance between two points is a straight line, but what will that distance be? How much time will he save by cutting across?

SOLUTION

Let's begin by drawing a picture to represent this situation and finding the length of the diagonal path.



The diagonal path forms a triangle inside the rectangle. We know this is a right triangle because the angle in the rectangle is a right angle. So the Pythagorean theorem can be used to find the length of the diagonal path, d.

$$\begin{aligned} 350^2 + 520^2 &= d^2 \\ 122{,}500 + 270{,}400 &= d^2 \\ 392{,}900 &= d^2 \\ \sqrt{392{,}900} &= \sqrt{d^2} \\ 626.8 \text{ ft} &\approx d \end{aligned}$$

There are two square roots of 16 and two solutions to the equation $x^2 = 16$, 4 and -4. In general, when you take the square root of a number, both the positive and negative results should be stated unless one does not make sense in the context of the problem.

In comparison, the sidewalk route would involve walking 350 feet + 520 feet = 870 feet. The distance John will save by cutting across the quad is 870 feet − 626.8 feet ≈ 243.2 feet.

To determine how much time John will save by cutting across the quad, we need to know how fast he walks. Let's assume that he walks about 3 feet per second. Since distance is rate times time, we can find the time by dividing the distance by the rate.

$$\begin{aligned} \text{Time} &= \frac{\text{distance}}{\text{rate}} \\ &= \frac{243.2 \text{ ft}}{3 \text{ ft/sec}} \\ &\approx 81.1 \text{ sec} \\ &\approx 1.35 \text{ min} \end{aligned}$$

John will save 1.35 minutes by cutting across the quad.

HOW IT WORKS

To find the length of the hypotenuse of a right triangle using the Pythagorean theorem:

1. Substitute the lengths of the legs into the equation $\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$.
2. Simplify the left side of the equation.
3. Take the square root of both sides.

EXAMPLE: If the lengths of the legs of a right triangle are 7 cm and 9 cm, find the length of the hypotenuse.

The solution involves writing and solving a quadratic equation.

$\text{leg}^2 + \text{leg}^2 = \text{hyp}^2$	Substitute the leg lengths into the equation
$7^2 + 9^2 = \text{hyp}^2$	Square the leg lengths
$49 + 81 = \text{hyp}^2$	Add
$130 = \text{hyp}^2$	Square root
$\sqrt{130} = \text{hyp}$	Use your calculator to approximate the square root
$11.4 \text{ cm} \approx \text{hyp}$	

To find the length of a leg of a right triangle using the Pythagorean theorem:

1. Substitute the lengths of the known leg and the hypotenuse into the equation $\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$.
2. Simplify both sides of the equation.
3. Isolate the square of the leg.
4. Take the square root of both sides.

EXAMPLE: If one leg of a right triangle is 4 inches long and the hypotenuse is 6 inches long, find the length of the other leg.

The solution involves writing and solving a quadratic equation. To isolate the length of the leg, undo the operations done to the length of the leg in the reverse order they were applied.

$\text{leg}^2 + \text{leg}^2 = \text{hyp}^2$	Substitute the lengths into the equation
$4^2 + \text{leg}^2 = 6^2$	Square the lengths
$16 + \text{leg}^2 = 36$	Subtract 16 from each side
$\text{leg}^2 = 20$	Square root
$\text{leg} = \sqrt{20}$	Use your calculator to approximate the square root
$\text{leg} \approx 4.47 \text{ in.}$	

2. a. Find the length of the hypotenuse of a right triangle if the legs each measure 10 feet.

$a^2+b^2=c^2$

$10^2+10^2=c^2$

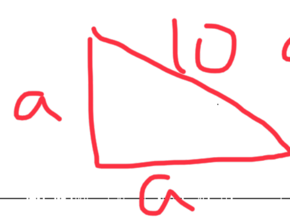
$100+100=c^2$

$\sqrt{200}=\sqrt{c^2}$

$c=\sqrt{200}$ (sqrt 200)

$c=\boxed{14.1 \text{ ft}}$

b. Find the length of the legs of a right triangle if the legs are the same length and the hypotenuse measures 10 feet.



$a^2+a^2=10^2$

$2a^2=100$

$a^2=50$

$\boxed{\sqrt{50} \quad 7.07 \text{ ft}}$

LOOK IT UP

Pythagorean Triple

When three whole numbers satisfy the Pythagorean theorem, they are known as a **Pythagorean triple.**

FOR EXAMPLE, 3, 4, and 5 form a Pythagorean triple. Pythagorean triples are always listed with the numbers in increasing order. The last number is the length of the hypotenuse.

3. a. Use the Pythagorean theorem to complete each of the following Pythagorean triples:

3, 4, 5 ← hyp
6, 8, 10
9, 12, 15
30, 40, 50
57, 76, 95
60, 80, 100

doubled

b. What relationship do you see between the first and second Pythagorean triples? Between the first and third?

tripled

Notice that if *a*, *b*, *c* is a Pythagorean triple, then so is every multiple of *a*, *b*, *c* that results in whole numbers.

4. a. If 5, 12, 13 is a Pythagorean triple, name two more triples related to it.

10, 24, 26   15, 36, 39

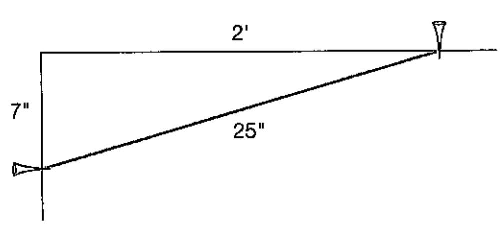
b. Complete the following two Pythagorean triples:

8, 15, 17
14, 48, 50

The Pythagorean theorem says that if a triangle is a right triangle, then the lengths of its sides satisfy the equation $\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$. The converse of this statement is also true. That is, if the lengths of the sides of a triangle satisfy the equation $\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$, then the sides form a right triangle.

EXAMPLE 2

A home builder is trying to square the corners of a property. A string is secured 7 inches out from the corner on one side and 2 feet out on the other side as shown in the diagram.



One side is angled until the two marks are exactly 25 inches apart, and then a stake is planted at the property corner and at each mark. Why do these measurements guarantee a square corner? What are some other measurements the builders could use that might be more convenient?

SOLUTION

We can begin by converting the 2 feet to 24 inches so that all three sides of the triangle are measured in the same unit of inches. Then we can substitute the three side lengths into the Pythagorean theorem to see if a true statement results.

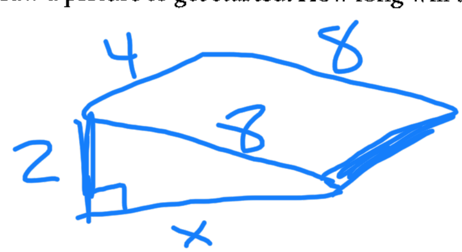
$$7^2 + 24^2 \stackrel{?}{=} 25^2$$
$$49 + 576 \stackrel{?}{=} 225$$
$$625 = 625$$

Since these values satisfy the Pythagorean theorem, the triangle is a right triangle and the corner is a right angle or "square." Any Pythagorean triple will work to square the corner. 3 feet, 4 feet, 5 feet is a commonly used Pythagorean triple that could work in this application.

Connect

5. Jack has instructions for building a skateboard ramp, and he needs to build a triangular frame on which to nail the ramp board. A 4 foot by 8 foot piece of plywood will serve as the ramp, and Jack wants the plywood oriented so that the ramp is as long as possible. The instructions say to choose the desired height for the ramp and then use the Pythagorean theorem to find the length of the frame that will sit on the ground. He wants the end of the ramp to be 2 feet off the ground.

Draw a picture to get started. How long will the bottom length of the frame be?



$2^2 + x^2 = 8^2$

$x = \sqrt{60}$

$x = 7.75$ ft

Reflect

WRAP-UP

What's the point?

The Pythagorean theorem can be used to find the length of one side of a right triangle when you know the lengths of the other two sides. This theorem has applications in geometry, construction, and other areas of mathematics.

What did you learn?

How to use the Pythagorean theorem to find the length of a side in a right triangle

How to solve problems using the Pythagorean theorem